

Rings with Effects

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Abstract

An e-ring is a pair (R, E) consisting of an associative ring R with unity 1 together with a subset $E \subseteq R$ of elements, called effects, with properties suggested by the so-called effect operators on a Hilbert space. We establish the basic properties of e-rings, investigate commutative e-rings called c-rings, relate certain c-rings called b-rings to Boolean algebras, and prove a structure theorem for b-rings.

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1 Introduction

Let \mathfrak{H} be a Hilbert space. In what follows, $\mathbb{B}(\mathfrak{H})$ denotes the $*$ -algebra of all bounded linear operators on \mathfrak{H} , and $\mathbb{G}(\mathfrak{H}) \subseteq \mathbb{B}(\mathfrak{H})$ is the subgroup of the additive group of $\mathbb{B}(\mathfrak{H})$ consisting of all Hermitian operators on \mathfrak{H} . The additive group $\mathbb{G}(\mathfrak{H})$, organized into a partially ordered abelian group as usual, will be called the *Hermitian group* for \mathfrak{H} . The identity operator $\mathbf{1}$ belongs to $\mathbb{G}(\mathfrak{H})$ and satisfies $\mathbf{0} \leq \mathbf{1}$. We define $\mathbb{E}(\mathfrak{H}) := \{E \in \mathbb{G}(\mathfrak{H}) \mid \mathbf{0} \leq E \leq \mathbf{1}\}$ and (following G. Ludwig [21]) we refer to operators in $\mathbb{E}(\mathfrak{H})$ as *effect operators* on \mathfrak{H} . We also define $\mathbb{P}(\mathfrak{H}) := \{P \in \mathbb{G}(\mathfrak{H}) \mid P^2 = P\}$ to be the set of all (orthogonal) *projection operators* on \mathfrak{H} . Then we have

$$\mathbf{0}, \mathbf{1} \in \mathbb{P}(\mathfrak{H}) \subseteq \mathbb{E}(\mathfrak{H}) \subseteq \mathbb{G}(\mathfrak{H}) \subseteq \mathbb{B}(\mathfrak{H}).$$

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In mathematical physics, the representation of observables by so-called POV-measures, i.e., $\mathbb{E}(\mathfrak{H})$ -valued measures on σ -fields [2], as well as by the more conventional $\mathbb{P}(\mathfrak{H})$ -valued measures, has now become a commonplace. Consequently, $\mathbb{E}(\mathfrak{H})$, $\mathbb{P}(\mathfrak{H})$, and suitable generalizations thereof, have come to be employed as algebraic models for the semantics of both sharp and unsharp quantum logics [5, 6, 8, 12, 15, 16, 20]. Thus motivated, we take the pair $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$ consisting of the ring $\mathbb{B}(\mathfrak{H})$ and the “effect algebra” $\mathbb{E}(\mathfrak{H})$ as a prototype for the following more general notion of an “e-ring.”

1.1 Definition. An *e-ring* is a pair (R, E) consisting of an associative ring R with unity 1 and a subset $E \subseteq R$ of elements called *effects* such that $0, 1 \in E$; $e \in E \Rightarrow 1 - e \in E$; and the set E^+ consisting of all finite sums $e_1 + e_2 + \cdots + e_n$ with $e_1, e_2, \dots, e_n \in E$ satisfies the following conditions: For all $a, b \in E^+$;

- (i) $-a \in E^+ \Rightarrow a = 0$,
- (ii) $1 - a \in E^+ \Rightarrow a \in E$,
- (iii) $ab = ba \Rightarrow ab \in E^+$,
- (iv) $aba \in E^+$,
- (v) $aba = 0 \Rightarrow ab = ba = 0$, and
- (vi) $(a - b)^2 \in E^+$.

The notion of an e-ring is mathematically equivalent to the notion of an effect-ordered ring originally introduced in [9], but reformulated to emphasize the role of the “effect algebra” E .

It is easy to see that the prototypic pair $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$ is an e-ring as soon as one observes that $\mathbb{E}(\mathfrak{H})^+$ is precisely the set of all positive Hermitian operators on \mathfrak{H} . (By a slight abuse of language, we call a Hermitian operator A “positive” if $A \geq 0$.) Indeed, every effect operator is positive by definition, and a finite sum of positive Hermitian operators is positive. Conversely, if A is a positive Hermitian operator, $\|A\|$ is the uniform operator norm of A , and n is a positive integer with $n \geq \|A\|$, then $(1/n)A$ is an effect operator, and A is a sum of n effect operators, each equal to $(1/n)A$.

In addition to the prototypic example $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$, we have the following simple examples of e-rings. (More examples will be given later.) We denote the set of positive integers by $\mathbb{N} := \{1, 2, 3, \dots\}$.

1.2 Example. Let R be a ring with unity 1 such that $n \cdot 1 \neq 0$ for all $n \in \mathbb{N}$, and define $E := \{0, 1\}$. Then (R, E) is an e-ring.

1.3 Example. If R is any subfield of the totally ordered field \mathbb{R} of real numbers and $E := \{e \in R \mid 0 \leq e \leq 1\}$, then (R, E) is an e-ring.

As Examples 1.2 and 1.3 illustrate, the definition of an e-ring (R, E) does not necessarily provide a strong connection between the algebraic structure of the set E of effects and the ring structure of R . For instance, if (R, E) is an e-ring and \tilde{R} is any extension ring of R such that $1 \cdot x = x \cdot 1 = x$ for all $x \in \tilde{R}$, then (\tilde{R}, E) is also an e-ring. The rather weak connection between E and R does not concern us here because, motivated by quantum measurement theory and quantum logic, we are mainly interested in the structure of the “effect algebra” E , and the enveloping ring R is just a convenient environment in which to study this structure.

Section 2 below, which treats the basic properties of e-rings, culminates with a theorem that the set P of projections in an e-ring acquires (at least) the structure of an orthomodular poset (Theorem 2.15) and a theorem stating that P indexes a so-called compression base for the directed group generated by the effects (Theorem 2.18). Section 3, which treats commutativity and coexistence in the effect algebra E of an e-ring, culminates in a structure theorem for a class of e-rings (called b-rings) in which every effect is a projection (Theorem 3.16). In a subsequent paper, we shall study square roots and polar decompositions in e-rings.

2 Basic Properties of e-Rings

We begin by extracting from an e-ring (R, E) an analogue G of the Hermitian group $\mathbb{G}(\mathfrak{H})$ for the prototype $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$.

2.1 Theorem. *Let (R, E) be an e-ring and let E^+ be the set of all sums of finite sequences of elements of E . Then*

$$G := E^+ - E^+ = \{a - b \mid a, b \in E^+\}$$

is a subgroup of the additive group of the ring R , and G is a directed partially ordered abelian group with positive cone $E^+ = \{g \in G \mid 0 \leq g\}$. Moreover, $E \subseteq G$ and E generates the group G .

Proof. Since E^+ is closed under addition, it is clear that $G = E^+ - E^+$ is a subgroup of the additive group of the ring R . Also, $0 \in E \subseteq E^+ \subseteq G$, and by Definition 1.1 (i), if both a and $-a$ belong to E^+ , then $a = 0$. Therefore, G is a partially ordered abelian group with positive cone E^+ , the partial order being given by $g \leq h \Leftrightarrow h - g \in E^+$ for $g, h \in G$ [14, p. 3]. By the

definition of G , every element $g \in G$ can be written in the form $g = a - b$ with $a, b \in E^+$, i.e., G is directed [14, p. 4]. Thus, E^+ generates the group G , and since E generates E^+ as an additive semigroup, it follows that E generates G as a group. \square

If (R, E) is an e-ring then, *as an additive abelian group*, R is partially ordered (but not necessarily directed) with E^+ as its positive cone; however, unless R is commutative, it is not necessarily a partially ordered ring (as usually understood) because E^+ need not be closed under multiplication.

2.2 Definition. Let (R, E) be an e-ring and let E^+ be the set of all sums of finite sequences of elements of E . Then:

- (i) The partially ordered additive abelian group $G := E^+ - E^+$ (Theorem 2.1) is called the *directed group* of (R, E) .
- (ii) Idempotent elements $p = p^2 \in G$ are called *projections*.

For the prototype e-ring $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$, the directed group is the Hermitian group $\mathbb{G}(\mathfrak{H})$, and $\mathbb{P}(\mathfrak{H})$ is the set of projections. In Example 1.2, the directed group $G = \{n \cdot 1 \mid n \in \mathbb{Z}\}$ of (R, E) is isomorphic to the totally ordered additive group of the ring \mathbb{Z} of integers. In Example 1.3, G is the additive subgroup of the field R with the total order inherited from \mathbb{R} . In both Examples 1.2 and 1.3, the only projections are 0 and 1.

The e-ring $(\mathbb{B}(\mathfrak{H}), \mathbb{E}(\mathfrak{H}))$ is a special case (when A is a type-I von Neumann factor) of the e-ring (A, E) in the following example.

2.3 Example. Let A be a unital C^* -algebra and let

$$E := \{aa^* \mid a \in A \text{ and } \exists b \in A, aa^* + bb^* = 1\}.$$

Then (A, E) is an e-ring, $E^+ = \{aa^* \mid a \in A\}$, and the directed group G for (A, E) is the additive group of self-adjoint elements in A .

In the sequel, we assume that (R, E) is an e-ring, 1 is the unity element in R , E^+ is the set of all sums of finite sequences of elements of E , $G = E^+ - E^+$ is the directed group of (R, E) , G is partially ordered with positive cone E^+ , and

$$P = \{p \in G \mid p = p^2\}$$

is the set of all projections in G . It is understood that P and E are partially ordered by the restrictions of the partial order \leq on G .

2.4 Lemma. *Let $g, h \in G$ and let $p \in P$. Then:*

- (i) $g^2 \in E^+$. (ii) $gh + hg \in G$. (iii) $g \in E^+ \Rightarrow ghg \in G$.
- (iv) $php \in G$. (v) $h \in E^+ \Rightarrow php \in E^+$.

Proof. (i) By hypothesis, G is directed, hence g is a difference of two elements in E^+ , so $g^2 \in E^+$ by Definition 1.1 (vi).

(ii) By (i), $gh + hg = (g + h)^2 - g^2 - h^2 \in G$.

(iii) As G is directed, there exist $a, b \in E^+$ such that $h = a - b$. By Definition 1.1 (iv), $gag, gbg \in E^+$, whence $ghg = gag - gbg \in G$.

(iv) As $p = p^2 \in G$, (i) implies that $p \in E^+$; hence (iv) follows from (iii).

(v) As in (iv), $p \in E^+$, so (v) follows from Definition 1.1 (iv). \square

2.5 Lemma. (i) $E = \{e \in G \mid 0 \leq e \leq 1\}$. (ii) $0, 1 \in P$. (iii) $p \in P \Rightarrow 1 - p \in P$. (iii) $P \subseteq E$.

Proof. (i) By Definition 1.1, we have $0, 1 \in E$. Evidently, if $e \in E$, then $e, 1 - e \in E \subseteq E^+ = \{g \in G \mid 0 \leq g\}$, whence $e \in G$ with $0 \leq e \leq 1$. Conversely, suppose $e \in G$ with $0 \leq e \leq 1$. Then $e, 1 - e \in E^+$, and it follows from Definition 1.1 (ii) that $e \in E$. Thus, $E = \{e \in G \mid 0 \leq e \leq 1\}$.

(ii) $0 = 0^2 \in G$ and $1 = 1^2 \in G$.

(iii) If $p \in G$ with $p = p^2$, then $1 - p \in G$ with $(1 - p)^2 = 1 - 2p + p^2 = 1 - p$.

(iv) Suppose $p \in P$. By Lemma 2.4 (i), $p = p^2 \in E^+$, i.e., $0 \leq p$. Thus, by (iii), $0 \leq 1 - p$, whence $0 \leq p \leq 1$, so $p \in E$ by (i). \square

In view of Lemma 2.5 (i), we shall refer to E as the *unit interval* in G . We have

$$0, 1 \in P \subseteq E \subseteq E^+ \subseteq G \subseteq R.$$

Equipped with the partially defined binary operation \oplus obtained by restricting $+$ on G to E , the unit interval E forms a so-called *effect algebra* [1]. The effect algebras arising from e-rings in this way are rather special in that they admit a (perhaps only partial) multiplicative structure (cf. [7]).

2.6 Lemma. *Let $d, e, f \in E$ with $ef = fe$. Then:*

- (i) $0 \leq ef \leq e, f \leq 1$. (ii) $0 \leq ede \leq e^2 \leq e \leq 1$.
- (iii) $0 \leq e, f \leq e + f - ef \leq 1$. (iv) $0 \leq e - e^2 \leq e, 1 - e \leq 1$.

Proof. Assume the hypotheses.

(i) By Definition 1.1 (iii), $0 \leq ef$. Likewise, $0 \leq e(1 - f) = e - ef$, so $ef \leq e$, and by symmetry, $ef \leq f$.

(ii) By Definition 1.1 (iv), $0 \leq ede$ and $0 \leq e(1-d)e = e^2 - ede$. Also, by (i) with $f = e$, we have $e^2 \leq e \leq 1$.
 (iii) By (i), $0 \leq f - ef$, so $e \leq e + f - ef$, and by symmetry, $f \leq e + f - ef$. Also, by Definition 1.1 (iii), $0 \leq (1-e)(1-f) = 1 - e - f + ef$, whence $e + f - ef \leq 1$.
 (iv) By (ii), $0 \leq e - e^2$. Also, by Lemma 2.4 (i), $0 \leq e^2$, whence $e - e^2 \leq e$. Finally, by (iii) with $f = e$, we have $2e - e^2 \leq 1$, so $e - e^2 \leq 1 - e$. \square

2.7 Lemma. *Let $g, h, k \in E^+$, $p \in P$, and $n \in \mathbb{N}$. Then: (i) $gh = 0 \Rightarrow hg = 0$. (ii) If $gk = kg$ and $hk = kh$, then $g \leq h \Rightarrow gk \leq hk$. (iii) If $gh = hg$, then $g \leq h \Rightarrow g^2 \leq h^2$. (iv) $g \leq np \Rightarrow g = gp = pg$. (v) $g^n = 0 \Rightarrow g = 0$.*

Proof. (i) $gh = 0 \Rightarrow ghg = 0 \Rightarrow hg = 0$ by Definition 1.1 (v).

(ii) Assume the hypotheses. Then $h - g \in E^+$ and $hk - gk = (h - g)k = k(h - g) \in E^+$ by Definition 1.1 (iii).

(iii) Assume the hypotheses. By (ii), $g^2 \leq gh$ and $gh \leq h^2$, so $g^2 \leq h^2$.

(iv) Assume the hypotheses. Then $g, np - g \in E^+$, whence $(1-p)g(1-p), (1-p)(np - g)(1-p) = -(1-p)g(1-p) \in E^+$ by Lemma 2.4 (v). Therefore, $(1-p)g(1-p) = 0$ by Definition 1.1 (i), and it follows from Definition 1.1 (v) that $(1-p)g = g(1-p) = 0$, i.e., $g = pg = gp$.

(v) We may assume that n is the smallest positive integer such that $g^n = 0$. If n is even and $k = n/2$, we have $g^k \cdot 1 \cdot g^k = 0$, so $g^k = g^k \cdot 1 = 0$ by Definition 1.1 (v), contradicting our assumption. Therefore n is odd. If $n = 1$, we are done, so we may assume that $n = 2k + 1$ where $k \in \mathbb{N}$. Then $g^k g g^k = 0$, so $g^{k+1} = g^k g = 0$ by Definition 1.1 (v), again contradicting our assumption. \square

According to part (i) of the following lemma, 1 is a so-called *order unit* in G [14, p. 4].

2.8 Lemma. (i) *If $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leq n \cdot 1$.* (ii) *If $a_1, a_2, \dots, a_n \in E^+$ and $a_1 + a_2 + \dots + a_n = 0$, then $a_1 = a_2 = \dots = a_n = 0$.*

Proof. (i) Write $g = a - b$ with $0 \leq a, b$. Then $0 \leq b = a - g$, whence $g \leq a$. As $a \in E^+$, there exist $e_1, e_2, \dots, e_n \in E$ with $a = e_1 + e_2 + \dots + e_n$. By Lemma 2.5 (i), $e_i \leq 1$ for $i = 1, 2, \dots, n$, and it follows that $g \leq a \leq n \cdot 1$.

(ii) Assume the hypotheses. It will be sufficient to prove that $a_1 = 0$. But, $-a_1 = a_2 + \dots + a_n \in E^+$, so $a_1 = 0$ by Definition 1.1 (i). \square

2.9 Theorem. *Let $e \in E$ and $p \in P$. Then the following conditions are mutually equivalent: (i) $e \leq p$, (ii) $e = ep = pe$, (iii) $e = pep$, (iv) $e = ep$, (v) $e = pe$.*

Proof. (i) \Rightarrow (ii). Assume that $e \leq p$ and let $d := p - e$. Then $e, d \in E^+$, $e + d = p$, and

$$(1 - p)e(1 - p) + (1 - p)d(1 - p) = (1 - p)p(1 - p) = 0.$$

By Lemma 2.5 (iii), $1 - p \in P$, whence by Lemma 2.4 (v),

$$(1 - p)e(1 - p), (1 - p)d(1 - p) \in E^+,$$

and it follows from Lemma 2.8 (ii) that $(1 - p)e(1 - p) = (1 - p)d(1 - p) = 0$. Therefore, by Definition 1.1 (v), $(1 - p)e = e(1 - p) = 0$, i.e., $e = pe = ep$.

(ii) \Rightarrow (iii) \Rightarrow (iv). Follows from $p = p^2$.

(iv) \Leftrightarrow (v). By Lemma 2.7 (i), $e = pe \Rightarrow (1 - p)e = 0 \Rightarrow e(1 - p) = 0 \Rightarrow e = ep$, and the converse implication follows by symmetry.

(v) \Rightarrow (i). Assume (v). Since (iv) \Leftrightarrow (v), we have $pe = ep = e$, so $(1 - e)p = p(1 - e) = p - e$, whence $p - e \in E^+$ by Definition 1.1 (iii), and therefore $e \leq p$. \square

2.10 Corollary. *Let $e \in E$ and $p \in P$. Then the following conditions are mutually equivalent: (i) $p \leq e$, (ii) $p = ep = pe$, (iii) $p + pep = pe + ep$, (iv) $p = ep$, (v) $p = pe$.*

Proof. Replace e by $1 - e$ and p by $1 - p$ in Theorem 2.9, noting that $p \leq e \Leftrightarrow 1 - e \leq 1 - p$. \square

2.11 Theorem. *Let $p, q \in P$. Then the following conditions are mutually equivalent: (i) $p + q \in E$, (ii) $p + q \leq 1$, (iii) $pq = 0$, (iv) $pq = qp = 0$, (v) $p + q \in P$.*

Proof. (i) \Rightarrow (ii) Follows from Lemma 2.5 (i).

(ii) \Rightarrow (iii). If $p + q \leq 1$, then $p \leq 1 - q$, and it follows from Theorem 2.9 that $p = p(1 - q) = p - pq$, whence $pq = 0$.

(iii) \Rightarrow (iv). Follows from Lemma 2.7 (i).

(iv) \Rightarrow (v). If $pq = qp = 0$, then $(p + q)^2 = p^2 + pq + qp + q^2 = p + q$.

(v) \Rightarrow (i). Follows from Lemma 2.5 (iv). \square

2.12 Theorem. *If $p, q \in P$, then the following conditions are mutually equivalent: (i) $pq \in P$, (ii) $pq \in E$, (iii) $pq = qp$, (iv) $pq = pqp$. Moreover, if any—hence all—of these conditions hold, then $pq = p \wedge q$ is the infimum (greatest lower bound) of p and q both in P and in E .*

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Assume that $e := pq \in E$. Since $e = pe = eq$, Theorem 2.9 implies that $e \leq p, q$ and $e = ep = qe$. Thus, $e^2 = epq = e$, so $e \in P$. As $e \leq p, q$, it follows that $p - e, q - e \in E^+$. Furthermore, $(p - e)(q - e) = pq - pe - eq + e^2 = 0$, whence Lemma 2.7 (i) implies that $0 = (q - e)(p - e) = qp - e$, i.e., $qp = e = pq$.

(iii) \Rightarrow (iv). If $pq = qp$, then $pq = p(pq) = pqp$.

(iv) \Rightarrow (i). If $pq = pqp$, then $(pq)^2 = pqpq = (pq)q = pq$, so $pq \in P$.

Suppose that any, hence all of the conditions (i)–(iv) hold. Then $pq \in P$ with $pq \leq p, q$. Also, if $e \in E$ with $e \leq p, q$, then $e = ep = eq$, whence $e(pq) = e$, i.e., $e \leq pq$. \square

2.13 Corollary. *If $p, q \in P$ and $pq = qp$, then $p + q - pq \in P$ and $p + q - pq = p \vee q$ is the supremum (least upper bound) of p and q both in P and in E .*

Proof. The mapping $r \mapsto (1 - r)$ is order inverting and of period 2 on P . Also $pq = qp \Leftrightarrow (1 - p)(1 - q) = (1 - q)(1 - p)$, and $1 - (1 - p)(1 - q) = p + q - pq$. \square

2.14 Corollary. *Let $p, q \in P$. Then $q - p \in E \Leftrightarrow p \leq q \Leftrightarrow q - p \in P$. Moreover, if $p \leq q$, then $q - p = q \wedge (1 - p)$ is the infimum of q and $1 - p$ both in P and in E .*

Proof. If $q - p \in E$, then $0 \leq q - p$, whence $p \leq q$. Suppose that $p \leq q$. Then $p = pq = qp$ by Theorem 2.9, so $q(1 - p) = (1 - p)q = q - p$ and it follows from Theorem 2.12 that $q - p \in P$ and that $q - p$ is the infimum both in E and P of q and $1 - p$. \square

2.15 Theorem. *With $p \mapsto 1 - p$ as orthocomplementation, P is an orthomodular poset (OMP) [6, 8] and, for $p, q \in P$ with $p \leq 1 - q$, the supremum in P of p and q is $p \vee q = p + q$.*

Proof. We have $0 \leq p \leq 1$ for all $p \in P$, and $p \mapsto 1 - p$ is an order-reversing mapping of period 2 on P . Let $p, q \in P$. If $p \leq 1 - q$, then $p + q \leq 1$, whence $p + q \in E$, so $p + q \in P$ with $pq = qp = 0$ by Theorem 2.11, and it follows from Corollary 2.13 that $p + q = p + q - pq$ is the supremum of p and q in P . Now suppose that $p \leq q$. By Corollary 2.14, $q - p \in P$, whence $q = p + (q - p)$ is the orthomodular identity. \square

2.16 Lemma. *Suppose that $d, e, f, d + e + f \in E$ with $d + e, d + f \in P$. Then $d, e, f \in P$.*

Proof. Assume the hypotheses and let $p := d + e \in P$ and $q := d + f \in P$. Then $p + f = d + e + f \leq 1$, so $f \leq 1 - p \in P$; hence by Theorem 2.9, $f = (1 - p)f = f - pf$, and therefore $pf = 0$. Also, $d \leq d + e = p$, so $pd = d$ by Theorem 2.9. Consequently, $pq = p(d + f) = d \in E$, and it follows from Theorem 2.12 that $d = pq = pq \in P$. As $d \leq p$ and $d, p \in P$, it follows from Corollary 2.14 that $e = p - d \in P$, and likewise $f = q - d \in P$. \square

The following theorem provides useful conditions—not directly involving multiplication—for determining whether an effect is a projection.

2.17 Theorem. *If $e \in E$, then the following conditions are mutually equivalent: (i) If $a, b, a + b \in E$, then $a, b \leq e \Rightarrow a + b \leq e$. (ii) If $d \in E$ with $d \leq e, 1 - e$, then $d = 0$. (iii) $e \in P$.*

Proof. (i) \Rightarrow (ii). Assume (i) and the hypotheses of (ii). Then $d + e \leq 1$ with $d, e \leq e$, whence $d + e \leq e$, and it follows that $d = 0$.

(ii) \Rightarrow (iii). Assume (ii). By Lemma 2.6 (iv), $0 \leq e - e^2 \leq e, 1 - e$, so $e - e^2 = 0$, i.e., $e = e^2$.

(iii) \Rightarrow (i). Suppose $e \in P$ and assume the hypotheses of (i). By Corollary 2.10, $a = ae$ and $b = be$, whence $(a + b)e = a + b$, and it follows that $a + b \leq e$. \square

If $p \in P$ and $g \in G$, then by Lemma 2.4 (iv), $pgp \in G$; hence we can define the mapping $J_p: G \rightarrow G$ by $J_p(g) = pgp$ for all $g \in G$. Thus, owing to Lemmas 2.4, 2.5, 2.16, and Theorem 2.11, we have the following theorem (see [10, 11]).

2.18 Theorem. *The family $(J_p)_{p \in P}$ is a compression base for G .*

The partially ordered abelian group G is said to be *archimedean* iff, whenever $g, h \in G$ and $ng \leq h$ for all $n \in \mathbb{N}$, it follows that $g \leq 0$ [14, p. 20]. An order-preserving group endomorphism $J: G \rightarrow G$ is called a *retraction* iff $J(1) \in E$ and, for all $e \in E$, $e \leq J(1) \Rightarrow J(e) = e$. If $p \in P$, it is clear that J_p is a retraction on G . Conversely, as a consequence of [9, Corollary 4.6], we have the following theorem.

2.19 Theorem. *If G is archimedean, then every retraction J on G has the form $J = J_p$ with $p = J(1) \in P$.*

3 Commuting Elements of G

We maintain our standing hypothesis that (R, E) is an e -ring, G is its directed group, and P is the OMP of projections in G .

3.1 Definition. Let $g, h \in G$. We write gCh iff $gh = hg$ and we define the *commutant* of g in G by $C(g) := \{h \in G \mid gCh\}$. More generally, if $X \subseteq G$, then $C(X) := \bigcap_{x \in X} C(x)$ is called the *commutant* of X .

In contrast with more-or-less standard usage, e.g., in operator theory, we use the notion of the commutant *only in relation to elements of G* , and not to general elements of the enveloping ring R .

If L is any OMP, then two elements $p, q \in L$ are said to be *Mackey compatible* iff there exist pairwise orthogonal elements $p_1, q_1, d \in L$ with $p = d \vee p_1$, and $q = d \vee q_1$ [8]. By Theorem 2.15, projections p, q in the OMP P are Mackey compatible iff there exist projections $p_1, q_1, d \in P$ with $d + p_1 + q_1 \leq 1$, $p = d + p_1$, and $q = d + q_1$. The next lemma provides a useful condition—not directly involving multiplication—for determining whether two projections commute.

3.2 Lemma. *If $p, q \in P$, then pCq iff p and q are Mackey compatible in P .*

Proof. Suppose that pCq . By Theorem 2.12, $pq \in P$ with $pq \leq p, q$; by Corollary 2.13, $p + q - pq \in P$; and by Corollary 2.14, $p_1 := p - pq \in P$ and $q_1 := q - pq \in P$. Thus, with $d := pq$, we have $d + p_1 + q_1 = p + q - pq \in P$ with $p = d + p_1$, and $q = d + q_1$.

Conversely, suppose there exist $p_1, q_1, d \in P$ such that $d + p_1 + q_1 \in P$, $p = d + p_1$, and $q = d + q_1$. Then $d + p_1 \leq d + p_1 + q_1 \leq 1$, whence dCp_1 by Theorem 2.11. Likewise, dCq_1 , and p_1Cq_1 , whence pCq . \square

In the following definition, the condition in Lemma 3.2 is generalized to effects $e, f \in E$.

3.3 Definition. Effects $e, f \in E$ are said to be *coexistent* iff there exist effects $d, e_1, f_1 \in E$ such that $d + e_1 + f_1 \in E$, $e = d + e_1$, and $f = d + f_1$.

The terminology “coexistent” is borrowed from the quantum theory of measurement [2]. (Some authors also refer to coexistent effects as being “Mackey compatible,” but, since coexistent effects need not commute, we prefer not to follow this practice.)

3.4 Lemma. *Let $e, f \in E$. Then: (i) If eCf , then e and f are coexistent. (ii) If $e + f \leq 1$, then e and f are coexistent.*

Proof. (i) Let $d := ef$, $e_1 := e - ef$, and $f_1 := f - ef$. By Lemma 2.6 (i), $d, e_1, f_1 \in E$. Also, $d + e_1 + f_1 = e + f - ef \in E$ by Lemma 2.6 (iii).

(ii) If $e + f \in E$, then $0 + e + f \in E$ with $e = 0 + e$ and $f = 0 + f$. \square

In general, the converse of Lemma 3.4 (i) is false. For instance, in the prototype $\mathbb{E}(\mathfrak{H})$, choose two effect operators A and B that do not commute. Then $\frac{1}{2}A$ and $\frac{1}{2}B$ are non-commuting effect operators; yet, since $\frac{1}{2}A + \frac{1}{2}B \leq \mathbf{1}$, they are coexistent. However, we do have the following result.

3.5 Theorem. *Let $p, q \in P$. Then, regarded as effects in E , the projections p and q are coexistent iff pCq .*

Proof. Combine Lemma 2.16 and Lemma 3.2. \square

In a Boolean algebra (i.e., a bounded complemented distributive lattice), every element has a unique complement; hence if an OMP is a Boolean algebra, the Boolean complementation mapping coincides with the orthocomplementation mapping. It is well known that an OMP is a Boolean algebra iff its elements are pairwise Mackey compatible [8]; hence we have the following.

3.6 Corollary. *The OMP P is a Boolean algebra iff $P \subseteq C(P)$. Moreover, if P is a Boolean algebra, then $p \mapsto 1 - p$ is the Boolean complementation mapping on P .*

In what follows, we shall be considering the condition $G \subseteq C(G)$ and the weaker condition $G \subseteq C(P)$. For instance, if the enveloping ring R is commutative, then $G \subseteq C(G)$. Since E generates the group G , it follows that $G \subseteq C(G) \Leftrightarrow E \subseteq C(E)$ and that $G \subseteq C(P) \Leftrightarrow E \subseteq C(P)$. Also, if $G \subseteq C(P)$, then $P \subseteq C(P)$, whence P is a Boolean algebra by Corollary 3.6. If the unital C^* -algebra A in Example 2.3 satisfies virtually any version of the spectral theorem (e.g., if A is a von Neumann algebra, or even an AW^* -algebra), then $P \subseteq C(P)$ will imply that $G \subseteq C(G)$.

Suppose that $G \subseteq C(G)$, let $g, h \in G$, and choose $a, b, c, d \in E^+$ such that $g = a - b$ and $h = c - d$. By Definition 1.1 (iii), $ac, ad, bc, bd \in E^+ \subseteq G$, whence $gh = ac - ad - bc + bd \in G$, and it follows that G is not only an additive abelian group, but a commutative subring of R . Clearly, with G thus organized into a ring, (G, E) is an e-ring, G is a partially ordered

commutative ring with unity 1, the partially ordered additive group G is the directed group of (G, E) , and (G, E) is a *c-ring* as per the following definition.

3.7 Definition. A *c-ring* is an e-ring (G, E) such that G is a commutative ring and $G = E^+ - E^+$.

If $G \subseteq C(G)$ and $R \neq G$, we can disregard the enveloping ring R and drop down to the *c-ring* (G, E) . Evidently, the passage from the e-ring (R, E) to the *c-ring* (G, E) affects neither the structure of the effect algebra E nor of the Boolean algebra P .

As a consequence of the Gelfand representation theorem [19, Theorem 4.4.3], the following example of a *c-ring* may be regarded as the commutative version of Example 2.3.

3.8 Example. Let X be a compact Hausdorff space, define $C(X, \mathbb{R})$ to be the ring of all continuous real-valued functions $f: X \rightarrow \mathbb{R}$ with pointwise operations, and let

$$E(X, \mathbb{R}) := \{e \in C(X, \mathbb{R}) \mid 0 \leq e(x) \leq 1, \forall x \in X\}.$$

Then $(C(X, \mathbb{R}), E(X, \mathbb{R}))$ is a *c-ring*, the partial order on $C(X, \mathbb{R})$ is the pointwise partial order, $C(X, \mathbb{R})$ is archimedean, and the Boolean algebra

$$P(X, \mathbb{R}) := \{p \in C(X, \mathbb{R}) \mid p(x) \in \{0, 1\}, \forall x \in X\}$$

of projections in $(C(X, \mathbb{R}), E(X, \mathbb{R}))$ consists of the characteristic set functions χ_K of compact open subsets K of X .

In the following example of a *c-ring*, the effects are “fuzzy subsets” of X in the sense of L. Zadeh [26].

3.9 Example. Let \mathcal{F} be a σ -field of subsets of a nonempty set X , define $\mathcal{B}(X, \mathcal{F}, \mathbb{R})$ to be the ring under pointwise operations of all bounded real-valued \mathcal{F} -measurable functions $f: X \rightarrow \mathbb{R}$, and let

$$\mathcal{E}(X, \mathcal{F}, \mathbb{R}) := \{e \in \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \mid 0 \leq e(x) \leq 1, \forall x \in X\}.$$

Then $(\mathcal{B}(X, \mathcal{F}, \mathbb{R}), \mathcal{E}(X, \mathcal{F}, \mathbb{R}))$ is a *c-ring*, the partial order on $\mathcal{B}(X, \mathcal{F}, \mathbb{R})$ is the pointwise partial order, $\mathcal{B}(X, \mathcal{F}, \mathbb{R})$ is archimedean, and the Boolean algebra

$$\mathcal{P}(X, \mathcal{F}, \mathbb{R}) := \{p \in \mathcal{B}(X, \mathcal{F}, \mathbb{R}) \mid p(x) \in \{0, 1\}, \forall x \in X\}$$

of projections in $\mathcal{E}(X, \mathcal{F}, \mathbb{R})$ consists of the characteristic set functions χ_M of sets $M \in \mathcal{F}$.

Recall that a partially ordered abelian group G is said to be *lattice ordered*, or for short, is an ℓ -group, iff every pair of elements $g, h \in G$ has an infimum $g \wedge_G h$ and a supremum $g \vee_G h$ in the partially ordered set G . The additive partially ordered abelian groups $C(X, \mathbb{R})$ and $\mathcal{B}(X, \mathcal{F}, \mathbb{R})$ in Examples 3.8 and 3.9 are ℓ -groups with pointwise minimum and maximum as the infimum and supremum, respectively.

If G has the property that, for every $a, b, c, d \in G$ with $a, b \leq c, d$ (i.e., $a \leq c, a \leq d, b \leq c$, and $b \leq d$), there exists $t \in G$ with $a, b \leq t \leq c, d$, then G has the *Riesz interpolation property*, or for short, G is an *interpolation group* [14, Chapter 2]. If G is an ℓ -group, then it is an interpolation group. (Just take t to be any element between $a \vee_G b$ and $c \wedge_G d$.) Thus, the directed groups $C(X, \mathbb{R})$ and $\mathcal{B}(X, \mathcal{F})$ are interpolation groups.

The so-called *MV-algebras*, which play an important role in the analysis of many-valued logics [3, 4] and in the classification of AF C^* -algebras [22], can be characterized as the effect algebras that are realized as unit intervals in abelian ℓ -groups with order units. Thus, the unit intervals $E(X, \mathbb{R})$ and $\mathcal{E}(X, \mathcal{F}, \mathbb{R})$ in Examples 3.8 and 3.9 are MV-algebras. Not every MV-algebra can be realized as the unit interval in a c-ring, but the author does not know a perspicuous characterization of those that can.

In the theory of operator algebras, there are well-known connections between commutativity and lattice structure. For instance, by a theorem of S. Sherman [25], a unital C^* -algebra A (Example 2.3) is commutative iff the directed group G of self-adjoint elements in A is an ℓ -group. On the other hand, by a result of R. Kadison [18], if A is a von Neumann algebra, then A is a factor iff the directed group G is an antilattice (i.e., only pairs of comparable elements can have an infimum or a supremum in G). Under suitable hypotheses (borrowed from the theory of operator algebras) similar results can be obtained for groups with compression bases [13]; hence for e-rings. If P is a Boolean algebra, then it is a lattice, so Corollary 3.6 already furnishes a hint of the commutativity-lattice connection for e-rings; further evidence is provided by Theorems 3.10, 3.11, and 3.12 below.

3.10 Theorem. *Suppose that G is an ℓ -group (or more generally, an interpolation group [14]). Then, (i) $G \subseteq C(P)$ and (ii) P is a Boolean algebra.*

Proof. Assume that G is an interpolation group.

(i) Let $0 \leq g \in G$ and $p \in P$. As G is directed, it will be sufficient to prove that gCp . By Lemma 2.8 (i), there exists $n \in \mathbb{N}$ such that $0 \leq g \leq n \cdot 1$, whence $0 \leq g \leq np + n(1 - p)$. As G is an interpolation group, there exist $x, y \in G$ with $0 \leq x \leq np$, $0 \leq y \leq n(1 - p)$, and $g = x + y$ (see [14, Proposition 2.1 (b)]). Thus, by Lemma 2.7 (iv), $x = xp = px$ and $y = y(1 - p) = (1 - p)y$, and it follows that $yp = py = 0$ and $gp = pg = x$.

(ii) Follows from (i) and Corollary 3.6. \square

The c-ring in Example 3.9 satisfies the conditions in the following theorem. Of course, the c-ring in Example 3.8 satisfies condition (i), and it satisfies condition (ii) if X is basically disconnected (i.e., the closure of every open F_σ subset of X remains open).

3.11 Theorem. *Suppose that (i) $G \subseteq C(P)$, and (ii) for every $g \in G$, there exists $p \in P$ with $(1 - p)g \leq 0 \leq pg$. Then G is an ℓ -group.*

Proof. The proof is adapted from [14, Proposition 8.9]. Let $g, h \in G$. It will be sufficient to prove that the supremum $g \vee_G h$ exists in G . By hypothesis, there exists $p \in P$ such that $(1 - p)(g - h) \leq 0 \leq p(g - h)$. Put

$$s := pg + (1 - p)h = p(g - h) + h = (1 - p)(h - g) + g.$$

Evidently, $h, g \leq s$. Furthermore, if $k \in G$ with $g, h \leq k$, then by Lemma 2.7 (i), $pg \leq pk$ and $(1 - p)h \leq (1 - p)k$, whence $s \leq pk + (1 - p)k = k$. Therefore, $s = g \vee_G h$. \square

A subset $A \subseteq P$ is said to be *orthogonal* iff, for all $a, b \in A$, $a \neq b \Rightarrow ab = 0$. If A is an orthogonal subset of P , then by Corollary 2.13 and induction on n , the sum $p := a_1 + a_2 + \cdots + a_n$ of finitely many distinct elements $a_1, a_2, \dots, a_n \in A$ belongs to P and coincides with the supremum $p := a_1 \vee a_2 \vee \cdots \vee a_n$ both in P and in E .

3.12 Theorem. *The following conditions are mutually equivalent:*

- (i) P is a Boolean algebra and P generates the group G .
- (ii) $G \subseteq C(G)$ and, for each $g \in G$, there is a finite orthogonal set $A \subseteq P$ such that g is a linear combination with integer coefficients of the elements of A .
- (iii) G is an ℓ -group and $E = P$.

- (iv) G is an interpolation group and 1 is a minimal order unit in G .
- (v) E is a Boolean algebra with $e \mapsto 1 - e$ as the Boolean complementation mapping.
- (vi) $E = P$.

Proof. (i) \Rightarrow (ii). Assume (i). By Corollary 3.6, $P \subseteq C(P)$ and, since P generates G , it follows that $G \subseteq C(G)$. Let $g \in G$. Then there are projections $p_i \in P$, $1 \leq i \leq n$, and integer coefficients c_i such that $g = \sum_{i=1}^n c_i p_i$. Let B be the sub-Boolean algebra of P generated by p_i , $1 \leq i \leq n$. Since B is a finitely generated Boolean algebra, it is finite. Let A be the set of atoms (minimal nonzero elements) in B . Then, if $a, b \in A$ with $a \neq b$, we have $ab = ba = a \wedge b = 0$ (Theorem 2.12), so A is a finite orthogonal subset of P . Also, each element in B , and in particular each p_i , can be written as a sum of certain of the projections in A . Thus, by gathering terms, we can write $g = \sum_{i=1}^n c_i p_i = \sum_{a \in A} k_a a$ with integer coefficients k_a for all $a \in A$.

(ii) \Rightarrow (iii). Assume (ii). Then $G \subseteq C(G) \subseteq C(P)$. Let $g \in G$, and let A be a finite orthogonal subset of P such that $g = \sum_{a \in A} k_a a$ for integer coefficients k_a . Define $A_+ := \{a \in A \mid k_a > 0\}$, $A_- := \{a \in A \mid k_a < 0\}$, and $p := \sum_{a \in A_+} a$. Then $p \in P$, $a \in A_+ \Rightarrow pa = k_a a$, and $a \in A_- \Rightarrow pa = 0$. Thus, $pg = \sum_{a \in A_+} k_a a \geq 0$ and $(1 - p)g = g - pg = \sum_{a \in A_-} k_a a \leq 0$; hence G is an ℓ -group by Theorem 3.11. Now suppose $g \in E$. Then, if $a \in A_-$, we have $0 \leq ga = ag$, whence $0 \leq ga = k_a a \leq 0$, so $k_a a = 0$. Consequently, $g = \sum_{a \in A_+} k_a a$. Also, if $a \in A_+$, we have $a \leq k_a a = ga \leq a$ (Lemma 2.6 (i)), so $k_a a = a$. Consequently, $g = \sum_{a \in A_+} k_a a = \sum_{a \in A_+} a = p \in P$. Therefore, $E = P$.

(iii) \Rightarrow (iv). Assume (iii). Then G is an interpolation group. Suppose p is an order unit in G and $p \leq 1$. Then $0 \leq p$ and there exists $n \in \mathbb{N}$ such that $0 \leq 1 - p \leq np$. As $0 \leq p \leq 1$, we have $p \in E = P$; hence by Lemma 2.7 (iv), $1 - p = (1 - p)p = 0$, i.e., $p = 1$. Thus 1 is a minimal order unit in G .

(iv) \Rightarrow (v). Assume (iv). By Theorem 3.10, $G \subseteq C(P)$ and P is a Boolean algebra with $p \mapsto 1 - p$ as the Boolean complementation. It will be enough to show that $E = P$. Thus, let $e \in E$ and suppose that $d \in E$ with $d \leq e, 1 - e$. Then $1 - e \leq 1 - d$ and $e \leq 1 - d$. Adding the last two inequalities, we find that $1 \leq 2(1 - d)$. Thus, if $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leq n \cdot 1 \leq 2n(1 - d)$, and it follows that $1 - d$ is an order unit in

G . But $1 - d \leq 1$; hence, by hypothesis, $1 - d = 1$, i.e., $d = 0$. Consequently, $e \in P$ by Theorem 2.17, and we conclude that $E = P$.

(v) \Rightarrow (vi). Assume (v) and let $e \in E$. If $d \in E$ with $d \leq e, 1 - e$, then since $1 - e$ is the Boolean complement of e in the Boolean algebra E , it follows that $d = 0$; hence $e \in P$ by Theorem 2.17. Consequently, $E = P$.

(vi) \Rightarrow (i). Assume (vi) and let $p, q \in P = E$. Since E generates the group G , it will be sufficient by Corollary 3.6 to prove that pCq . Let $d := pqp$. By Lemma 2.6 (ii), $d \in E = P$, and it is clear that $d = dp = pd$, whence $d \leq p$. Thus, $dqd = dpqp d = d^3 = d$, and it follows that $d(1 - q)d = 0$. Therefore, $d(1 - q) = 0$, i.e., $d = dq$, so $d \leq q$. By Corollary 2.14, $p_1 := p - d \in P$ and $q_1 := q - d \in P$. Also, $pq_1p = pqp - pdp = d - d = 0$, so $pq_1 = 0$, $p = d + p_1$, $q = d + q_1$, and by Theorem 2.11 $d + p_1 + q_1 = p + q_1 \in P$. Consequently, p and q are Mackey compatible, so pCq by Lemma 3.2. \square

If the e-ring (R, E) satisfies any, hence all, of the conditions (i)–(v) in Theorem 3.12, then $G \subseteq C(G)$ by condition (ii), and we can drop down to the c-ring (G, E) , which of course will continue to satisfy conditions (i)–(v).

3.13 Definition. A *b-ring* is a c-ring (G, E) satisfying any, hence all, of the conditions (i)–(v) in Theorem 3.12.

The b-ring in the following example is a modification of Example 3.9 in which the totally ordered field \mathbb{R} is replaced by the totally ordered ring \mathbb{Z} of integers and the σ -field \mathcal{F} is replaced by any field of sets.

3.14 Example. Let \mathcal{F} be a field of subsets of a nonempty set X , define $\mathcal{B}(X, \mathcal{F}, \mathbb{Z})$ to be the ring under pointwise operations of all bounded functions $f: X \rightarrow \mathbb{Z}$ such that $f^{-1}(z) \in \mathcal{F}$ for all $z \in \mathbb{Z}$, and let

$$\mathcal{E}(X, \mathcal{F}, \mathbb{Z}) := \{e \in \mathcal{B}(X, \mathcal{F}, \mathbb{Z}) \mid e(x) \in \{0, 1\}, \forall x \in X\}.$$

Then $(\mathcal{B}(X, \mathcal{F}, \mathbb{Z}), \mathcal{E}(X, \mathcal{F}, \mathbb{Z}))$ is a b-ring, the partial order on $\mathcal{B}(X, \mathcal{F}, \mathbb{Z})$ is the pointwise partial order, and $\mathcal{B}(X, \mathcal{F}, \mathbb{Z})$ is archimedean. Thus the effects in $\mathcal{E}(X, \mathcal{F}, \mathbb{Z})$, which coincide with the projections for the b-ring $(\mathcal{B}(X, \mathcal{F}, \mathbb{Z}), \mathcal{E}(X, \mathcal{F}, \mathbb{Z}))$, are the characteristic set functions χ_M of sets $M \in \mathcal{F}$.

Under set-inclusion, a field \mathcal{F} of subsets of a nonempty set X is a Boolean algebra, and in Example 3.14, the Boolean algebra $\mathcal{E}(X, \mathcal{F}, \mathbb{Z})$ is isomorphic to \mathcal{F} . By the Stone representation theorem, every Boolean algebra B is isomorphic to the field \mathcal{F} of compact open subsets of a compact Hausdorff

totally-disconnected space X ; hence, *every Boolean algebra can be realized as the Boolean algebra of projections in a b-ring.*

The functions $f: X \rightarrow \mathbb{Z}$ in Example 3.14 can be regarded as “signed multisets” by thinking of $f(x)$ as the “signed multiplicity of x in f .” In [17], T. Hailperin suggests that, in contemporary algebraic terms, the true realization of Boole’s original ideas is not what is now called a Boolean algebra, but rather it is an algebra of signed multisets forming a commutative ring with unity and with no nonzero additive or multiplicative nilpotents. Our b-ring $(\mathcal{B}(X, \mathcal{F}, \mathbb{Z}), \mathcal{E}(X, \mathcal{F}, \mathbb{Z}))$ is precisely such an algebra, and the “b” in “b-ring” is meant to suggest this Boolean connection.

3.15 Theorem. *Let (G, E) and (H, F) be b-rings and let $\phi: E \rightarrow F$ be a Boolean homomorphism from the Boolean algebra E to the Boolean algebra F . Then ϕ admits a unique extension to a group homomorphism $\Phi: G \rightarrow H$ of the additive group G into the additive group H . Moreover, $\Phi: G \rightarrow H$ is an order-preserving ring homomorphism with $\Phi(1) = 1$.*

Proof. The Boolean homomorphism $\phi: E \rightarrow F$ preserves 0, 1, finite infima, and finite suprema. For $p, q \in E = P$, we have $p \wedge q = pq$; hence $\phi(pq) = \phi(p \wedge q) = \phi(p) \wedge \phi(q) = \phi(p)\phi(q)$, i.e., ϕ preserves products of projections. Also, if $p + q \in E$, then $p \vee q = p + q$; hence $\phi(p + q) = \phi(p \vee q) = \phi(p) \vee \phi(q) = \phi(p) + \phi(q)$, i.e., $\phi: E \rightarrow H$ preserves existing sums in E . Since G is an interpolation group, a theorem of S. Pulmannová [23] implies that ϕ admits a unique extension to a group homomorphism $\Phi: G \rightarrow H$. As $\phi(E) \subseteq F$, it follows that $\Phi(E^+) \subseteq F^+$, whence Φ is order preserving. Every element in G is a finite linear combination of projections with integer coefficients, and since Φ preserves products of projections, it follows that Φ preserves products. Obviously, $\Phi(1) = \phi(1) = 1$. \square

The following is the fundamental structure theorem for b-rings.

3.16 Theorem. *Let (G, E) be a b-ring, let X be the Stone space of the Boolean algebra E , and let \mathcal{F} be the field of compact open subsets of X . Then there is an order and ring isomorphism $\Phi: G \rightarrow \mathcal{B}(X, \mathcal{F}, \mathbb{Z})$ such that the restriction ϕ of Φ to E is a Boolean isomorphism of E onto $\mathcal{E}(X, \mathcal{F}, \mathbb{Z})$.*

Proof. The projections in $\mathcal{E}(X, \mathcal{F}, \mathbb{Z})$ are characteristic set functions χ_K of compact open subsets K of X ; hence by Stone’s representation theorem, there is a Boolean isomorphism $\phi: E \rightarrow \mathcal{E}(X, \mathcal{F}, \mathbb{Z})$. By Theorem 3.15, ϕ can be extended to an order-preserving ring homomorphism $\Phi: G \rightarrow \mathcal{B}(X, \mathcal{F}, \mathbb{Z})$

and $\phi^{-1}: \mathcal{E}(X, \mathcal{F}, \mathbb{Z}) \rightarrow E$ can be extended to an order-preserving ring homomorphism $\Psi: \mathcal{B}(X, \mathcal{F}, \mathbb{Z}) \rightarrow G$. The ring endomorphism $\Psi \circ \Phi: G \rightarrow G$ is the identity on E , and E generates G ; hence $\Psi \circ \Phi: G \rightarrow G$ is the identity on G . Likewise $\Phi \circ \Psi$ is the identity on $\mathcal{B}(X, \mathcal{F}, \mathbb{Z})$, so Φ is an order-preserving ring isomorphism with $\Psi = \Phi^{-1}$. \square

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